

Concurrence for multipartite states

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Abstract

We construct a generalized concurrence for general multipartite states based on local W-class and GHZ-class operators. We explicitly construct the corresponding concurrence for three-partite states. The construction of the concurrence is interesting since it is based on local operators.

1 Introduction

Concurrence is one of the most applied measure of entanglement. In recent years there have been some proposals to generalize this measure of entanglement to general multipartite states [1, 2]. Recently, we have also defined concurrence classes for multi-qubit mixed states based on an orthogonal complement of a positive operator valued measure (POVM) on quantum phase [3]. Moreover, we have constructed different concurrence classes for general pure multipartite states in [4]. In this paper, we will construct generalized concurrence for pure general multipartite states based on the complement of a POVM on quantum phase. However, this measure is not equal to our concurrence classes, where we have added these concurrence classes and then took the square root of them. But by rewriting our linear operators as sums and take the expectation value of each of these operators, we are able to construct a general formula for concurrence. We will consider a general, multipartite quantum system with m subsystems $\mathcal{Q} = \mathcal{Q}_m(N_1, N_2, \dots, N_m)$, denoting its general state as $|\Phi\rangle = \sum_{l_1=1}^{N_1} \cdots \sum_{l_m=1}^{N_m} \alpha_{l_1, l_2, \dots, l_m} |l_1, l_2, \dots, l_m\rangle$. Moreover, let $\rho_{\mathcal{Q}} = \sum_{n=1}^N p_n |\Phi_n\rangle \langle \Phi_n|$, for all $0 \leq p_n \leq 1$ and $\sum_{n=1}^N p_n = 1$, denote a density operator acting on the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the j th Hilbert space is given by $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. Finally, let us introduce a complex conjugation operator \mathcal{C}_m that acts on a general multipartite state $|\Phi\rangle$ as $\mathcal{C}_m |\Phi\rangle = \sum_{l_1=1}^{N_1} \cdots \sum_{l_m=1}^{N_m} \alpha_{l_1, l_2, \dots, l_m}^* |l_1, l_2, \dots, l_m\rangle$.

2 General multipartite states

In this section, we will construct concurrence for general pure multipartite states $\mathcal{Q}_m^p(N_1, \dots, N_m)$. In our construction, we will use linear operators that are constructed by the orthogonal complement of POVM on quantum phase [3, 4]. In order to simplify our presentation, we will use $\Lambda_m = k_1, l_1; \dots; k_m, l_m$ as an

abstract multi-index notation. In the m -partite case, the off-diagonal elements of the matrix corresponding to

$$\tilde{\Delta}_{\mathcal{Q}}(\varphi_{\mathcal{Q}_1;k_1,l_1}, \dots, \varphi_{\mathcal{Q}_m;k_m,l_m}) = \tilde{\Delta}_{\mathcal{Q}_1}(\varphi_{\mathcal{Q}_1;k_1,l_1}) \otimes \dots \otimes \tilde{\Delta}_{\mathcal{Q}_m}(\varphi_{\mathcal{Q}_m;k_m,l_m}), \quad (1)$$

where the orthogonal complement of our POVM

$$\Delta(\varphi_{\mathcal{Q}_j;k_j,l_j}) = \sum_{l_j,k_j=1}^{N_j} e^{i\varphi_{k_j,l_j}} |k_j\rangle\langle l_j| \quad (2)$$

is given by $\tilde{\Delta}_{\mathcal{Q}_j}(\varphi_{\mathcal{Q}_j;k_j,l_j}) = \mathcal{I}_{N_j} - \Delta_{\mathcal{Q}_j}(\varphi_{\mathcal{Q}_j;k_j,l_j})$. \mathcal{I}_{N_j} is the N_j -by- N_j identity matrix for subsystem j . $\tilde{\Delta}_{\mathcal{Q}}(\varphi_{\mathcal{Q}_1;k_1,l_1}, \dots, \varphi_{\mathcal{Q}_m;k_m,l_m})$ has phases that are sums or differences of phases originating from two and m subsystems. That is, in the latter case the phases of $\tilde{\Delta}_{\mathcal{Q}}(\varphi_{\mathcal{Q}_1;k_1,l_1}, \dots, \varphi_{\mathcal{Q}_m;k_m,l_m})$ take the form $(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2} \pm \dots \pm \varphi_{\mathcal{Q}_m;k_m,l_m})$ and identification of these joint phases makes our distinguishing possible. Thus, we can define linear operators for the W^m class which are sums and differences of phases of two subsystems, i.e., $(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}})$. That is, for the W^m class we have

$$\begin{aligned} \tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{W_{\Lambda_m}^m} &= \mathcal{I}_{N_1} \otimes \dots \otimes \tilde{\Delta}_{\mathcal{Q}_{r_1}}(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}^{\frac{\pi}{2}}) \\ &\quad \otimes \dots \otimes \tilde{\Delta}_{\mathcal{Q}_{r_2}}(\varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}^{\frac{\pi}{2}}) \otimes \dots \otimes \mathcal{I}_{N_m}. \end{aligned} \quad (3)$$

Next, we could write the linear operator $\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{W_{\Lambda_m}^m}$ as a direct sum of the upper and lower anti-diagonal

$$\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{W_{\Lambda_m}^m} = \mathcal{U}\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{W_{\Lambda_m}^m} + \mathcal{L}\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{W_{\Lambda_m}^m}. \quad (4)$$

The set of linear operators for the W^m classes gives the W^m class concurrence.

For the GHZ^m class, we define linear operators based on our POVM which are sums and differences of phases of m -subsystems, i.e., $(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}} \pm \dots \pm \varphi_{\mathcal{Q}_m;k_m,l_m})$. That is, for the GHZ^m class we have

$$\begin{aligned} \tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m} &= \tilde{\Delta}_{\mathcal{Q}_1}(\varphi_{\mathcal{Q}_1;k_1,l_1}^{\pi}) \otimes \dots \otimes \tilde{\Delta}_{\mathcal{Q}_{r_1}}(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}^{\frac{\pi}{2}}) \\ &\quad \otimes \dots \otimes \tilde{\Delta}_{\mathcal{Q}_{r_2}}(\varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}^{\frac{\pi}{2}}) \otimes \dots \otimes \tilde{\Delta}_{\mathcal{Q}_m}(\varphi_{\mathcal{Q}_m;k_m,l_m}^{\pi}). \end{aligned} \quad (5)$$

where by choosing $\varphi_{\mathcal{Q}_j;k_j,l_j}^{\pi} = \pi$ for all $k_j < l_j$, $j = 1, 2, \dots, m$, we get an operator which has the structure of the Pauli operator σ_x embedded in a higher-dimensional Hilbert space and coincides with σ_x for a single-qubit. There are $\frac{m(m-1)}{2}$ linear operators for the GHZ^m class.

Next, we write the linear operators for the GHZ^m class as

$$\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m} = \mathfrak{P}_1\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m} + \mathfrak{P}_2\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m} + \dots, \quad (6)$$

where the operators $\mathfrak{P}_i\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m}$ are constructed by pairing of elements of the POVM with sums and differences of quantum phases. For higher dimensional quantum systems, it is difficult to write $\tilde{\Delta}_{\mathcal{Q}_{r_1},r_2(N_{r_1},N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m}$ in terms

of $\mathfrak{P}_i \tilde{\Delta}_{\mathcal{Q}_{r_1, r_2}(N_{r_1}, N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m}$. However, we will give an explicit expression for general three-partite states in the next section. Moreover, we define the linear operators for the GHZ^{m-1} class of m -partite states based on our POVM which are sums and differences of phases of $m-1$ -subsystems, i.e., $(\varphi_{\mathcal{Q}_{r_1}; k_{r_1}, l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2}; k_{r_2}, l_{r_2}} \pm \dots \pm \varphi_{\mathcal{Q}_{m-1}; k_{m-1}, l_{m-1}} \pm \varphi_{\mathcal{Q}_{m-1}; k_{m-1}, l_{m-1}})$. That is, for the GHZ^{m-1} class we have

$$\begin{aligned} \tilde{\Delta}_{\mathcal{Q}_{r_1, r_2, r_3}(N_{r_1}, N_{r_2})}^{\text{GHZ}_{\Lambda_m}^{m-1}} &= \tilde{\Delta}_{\mathcal{Q}_{r_1}}(\varphi_{\mathcal{Q}_{r_1}; k_{r_1}, l_{r_1}}^{\frac{\pi}{2}}) \otimes \tilde{\Delta}_{\mathcal{Q}_{r_2}}(\varphi_{\mathcal{Q}_{r_2}; k_{r_2}, l_{r_2}}^{\frac{\pi}{2}}) \otimes \\ &\quad \tilde{\Delta}_{\mathcal{Q}_{r_3}}(\varphi_{\mathcal{Q}_{r_3}; k_{r_3}, l_{r_3}}^{\pi}) \otimes \dots \otimes \\ &\quad \tilde{\Delta}_{\mathcal{Q}_{m-1}}(\varphi_{\mathcal{Q}_{m-1}; k_{m-1}, l_{m-1}}^{\pi}) \otimes \mathcal{I}_{N_m}, \end{aligned} \quad (7)$$

where $1 \leq r_1 < r_2 < \dots < r_{m-1} < m$. Note that we need to write these operators also as direct sums as we did for GHZ^m class since they belong to the same operator class. Then, for a general pure state let

$$\begin{aligned} \mathcal{C}(\mathcal{Q}_{r_1, r_2}^{W^m}(N_{r_1}, N_{r_2})) &= \sum_{\forall k_j, l_j} \left(\left| \langle \Phi | \mathfrak{U} \tilde{\Delta}_{\mathcal{Q}_{r_1, r_2}(N_{r_1}, N_{r_2})}^{W^m} \mathcal{C}_m \Phi \rangle \right|^2 \right. \\ &\quad \left. + \left| \langle \Phi | \mathfrak{L} \tilde{\Delta}_{\mathcal{Q}_{r_1, r_2}(N_{r_1}, N_{r_2})}^{W^m} \mathcal{C}_m \Phi \rangle \right|^2 \right), \end{aligned} \quad (8)$$

$$\mathcal{C}(\mathcal{Q}_{r_1, r_2}^{\text{GHZ}^m}(N_{r_1}, N_{r_2})) = \sum_{\forall k_j, l_j} \sum_{i \geq m-2} \left| \langle \Phi | \mathfrak{P}_i \tilde{\Delta}_{\mathcal{Q}_{r_1, r_2}(N_{r_1}, N_{r_2})}^{\text{GHZ}_{\Lambda_m}^m} \mathcal{C}_m \Phi \rangle \right|^2 \quad (9)$$

and e.g.,

$$\mathcal{C}(\mathcal{Q}_{r_1, r_2, r_3}^{\text{GHZ}^{m-1}}(N_{r_1}, N_{r_2})) = \sum_{\forall k_j, l_j} \sum_{i \geq m-3} \left| \langle \Phi | \mathfrak{P}_i \tilde{\Delta}_{\mathcal{Q}_{r_1, r_2, r_3}(N_{r_1}, N_{r_2})}^{\text{GHZ}_{\Lambda_m}^{m-1}} \mathcal{C}_m \Phi \rangle \right|^2 \quad (10)$$

Then the concurrence is defined by adding these terms and the take square root of them as follows

$$\begin{aligned} \mathcal{C}(\mathcal{Q}_m^p(N_1, \dots, N_m)) &= (\mathcal{N}_m \{ \sum_{r_2 > r_1=1}^m \mathcal{C}(\mathcal{Q}_{r_1, r_2}^{W^m}(N_{r_1}, N_{r_2})) \\ &\quad + \sum_{r_2 > r_1=1}^m \mathcal{C}(\mathcal{Q}_{r_1, r_2}^{\text{GHZ}^m}(N_{r_1}, N_{r_2})) \\ &\quad + \sum_{r_3 > r_2 > r_1=1}^m \mathcal{C}(\mathcal{Q}_{r_1, r_2, r_3}^{\text{GHZ}^{m-1}}(N_{r_1}, N_{r_2})) + \dots \})^{1/2}, \end{aligned} \quad (11)$$

where \mathcal{N}_m is a normalization constant. Note that for three-partite states our concurrence consists of two parts $\mathcal{C}(\mathcal{Q}_{r_1, r_2}^{W^3}(N_{r_1}, N_{r_2}))$ and $\mathcal{C}(\mathcal{Q}_{r_1, r_2}^{\text{GHZ}^3}(N_{r_1}, N_{r_2}))$ which we will discuss in the next section. However, for four-partite states we have $\mathcal{C}(\mathcal{Q}_{r_1, r_2}^{W^3}(N_{r_1}, N_{r_2}))$, $\mathcal{C}(\mathcal{Q}_{r_1, r_2}^{\text{GHZ}^3}(N_{r_1}, N_{r_2}))$, and $\mathcal{C}(\mathcal{Q}_{r_1, r_2, r_3}^{\text{GHZ}^3}(N_{r_1}, N_{r_2}))$. Moreover, we can in principle define a concurrence for arbitrary multipartite states as

$$\mathcal{C}(\mathcal{Q}_m(N_1, \dots, N_m)) = \inf_{\Phi} \mathcal{C}(\mathcal{Q}_m^p(N_1, \dots, N_m)). \quad (12)$$

However, to evaluate it one needs to find a pure decomposition of density matrix of a given multipartite state which is a very difficult task.

3 General pure three-partite states

In this section, as an illustrative example, we will construct concurrence for pure three-partite quantum system $\mathcal{Q}_3^p(N_1, N_2, N_3)$ based on the orthogonal complement of our POVM. For three-partite states, we have two different joint phases in our POVM, those which are sums and differences of phases of two subsystems, i.e., $(\varphi_{\mathcal{Q}_1; k_1, l_1} \pm \varphi_{\mathcal{Q}_2; k_2, l_2})$ and those which are sums and differences of phases of three subsystems, i.e., $(\varphi_{\mathcal{Q}_1; k_1, l_1} \pm \varphi_{\mathcal{Q}_2; k_2, l_2} \pm \varphi_{\mathcal{Q}_3; k_3, l_3})$. The first one identifies the W^3 class operator and the second one identifies the GHZ^3 class operator. For the W^3 class, we have

$$\begin{aligned} \mathcal{C}(\mathcal{Q}_3^{W^3}(N_1, N_2, N_3)) = & \sum_{l_1 > k_1=1}^{N_1} \sum_{l_2 > k_2=1}^{N_2} \sum_{k_3=l_3=1}^{N_3} |\alpha_{k_1, l_2, k_3} \alpha_{l_1, k_2, l_3} - \alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3}|^2 \\ & + \sum_{l_1 > k_1=1}^{N_1} \sum_{l_3 > k_3=1}^{N_3} \sum_{k_2=l_2=1}^{N_2} |\alpha_{k_1, k_2, l_3} \alpha_{l_1, l_2, k_3} - \alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3}|^2 \\ & + \sum_{l_2 > k_2=1}^{N_2} \sum_{l_3 > k_3=1}^{N_3} \sum_{k_1=l_1=1}^{N_1} |\alpha_{k_1, k_2, l_3} \alpha_{l_1, l_2, k_3} - \alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3}|^2, \end{aligned} \quad (13)$$

and for the GHZ^3 class, we have

$$\begin{aligned} \mathcal{C}(\mathcal{Q}_3^{GHZ^3}(N_1, N_2, N_3)) = & \sum_{l_1 > k_1=1}^{N_1} \sum_{l_2 > k_2=1}^{N_2} \sum_{l_3 > k_3=1}^{N_3} [\\ & |\alpha_{k_1, l_2, l_3} \alpha_{l_1, k_2, k_3} - \alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3}|^2 + |\alpha_{k_1, l_2, k_3} \alpha_{l_1, k_2, l_3} - \alpha_{k_1, k_2, l_3} \alpha_{l_1, l_2, k_3}|^2 \\ & + |\alpha_{k_1, k_2, l_3} \alpha_{l_1, l_2, k_3} - \alpha_{k_1, l_2, l_3} \alpha_{l_1, k_2, k_3}|^2 + |\alpha_{k_1, l_2, k_3} \alpha_{l_1, k_2, l_3} - \alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3}|^2 \\ & + |\alpha_{k_1, l_2, k_3} \alpha_{l_1, k_2, l_3} - \alpha_{k_1, k_2, l_3} \alpha_{l_1, k_2, k_3}|^2 + |\alpha_{k_1, k_2, l_3} \alpha_{l_1, l_2, k_3} - \alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3}|^2]. \end{aligned} \quad (14)$$

Note that these expressions are not equal to our W class and GHZ class concurrences constructed in [4], where we have constructed our concurrences classes based on direct use of two class of operators. Thus the concurrence for a general pure three-partite state is give by

$$\begin{aligned} \mathcal{C}(\mathcal{Q}_3^p(N_1, N_2, N_3)) = & (\mathcal{N}_3[\mathcal{C}(\mathcal{Q}_3^{W^3}(N_1, N_2, N_3)) \\ & + \mathcal{C}(\mathcal{Q}_3^{GHZ^3}(N_1, N_2, N_3))])^{1/2}. \end{aligned} \quad (15)$$

This concurrence also coincides with the generalized concurrence for three-partite states[1]. Moreover, for m -partite states with $m \geq 3$, our concurrence is not the equal to concurrence tensor [5].

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